Repr. of Compact Lie groups. Bröcker + tom Dieck.

$S^1 = U(1) = SO(2)$ $\qquad$ $S^3 = Sp(1) = SU(2) = \widetilde{SO}(3)$

$\cap$ $\qquad$ $\cap$

$\mathbb{C} = \mathbb{R}^2$ $\qquad$ $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$

§ Integration (I.5)

G Lie group (Calculus + Linear algebra)

$$\left.\begin{array}{c} G_L \curvearrowright G \curvearrowleft G_R \\ hg \leftarrow g \mapsto g h^{-1} \end{array}\right\} \Rightarrow \Omega^k(G)^{G_L} \cong \Lambda^k \underbrace{\mathfrak{g}^*}_{T_e^*G} \cong \Omega^k(G)^{G_R}$$

Choose $dg \in \Omega^{TOP}(G)^{G_L} \simeq \Lambda^{TOP}\mathfrak{g}^* \simeq \mathbb{R}$

$\rightsquigarrow$ $\int : C(G) \longrightarrow \mathbb{R}$ $\qquad$ $\int_G f(g)\, dg$

$Vol(G) = \int_G dg = 1$ $\qquad$ ($\because$ G COMPACT)

$dg \in \Omega^{TOP}(G)^{G_L} \rightsquigarrow$ Left inv. integral.

Clam: Right inv. $(\Rightarrow$ Ad-inv.$)$

Proof: $\delta h \in \Omega^{TOP}(G)^{G_R}$

$$\int_g f(g)\,dg \overset{\int \delta h = 1}{=\!=} \int_h \left(\int_g f(g)\,dg \right) \delta h$$

$$\overset{\text{left inv.}}{=\!=} \int_h \left(\int_g f(gh)\,dg \right) \delta h$$

$$\overset{\text{Fubini}}{=\!=} \int_g \left(\int_h f(gh)\,\delta h \right) dg$$

$$\overset{\text{R-inv} + \int dg = 1}{=\!=} \int_h f(h)\,\delta h$$

QED.

Integration / Averaging

$$\frac{1}{\mathrm{Vol}(G)} \int_G (-)\, dg \quad \Big\| \quad \frac{1}{|G|} \sum_{g \in G} (\quad)$$

Compact Lie || finite

- $G \curvearrowright V / \mathbb{C} \Rightarrow \exists$ G-inv. inner product.

$\Rightarrow G \longrightarrow U(N)$ unitary.

$\Rightarrow$ complete reducibility, irred. $\equiv$ indec.

§ Character theory (II 2.3)

- $G \curvearrowright V$ determined by

  $\chi_V : G \longrightarrow GL(V) \xrightarrow{Tr} \mathbb{C}$ character

  $\chi_V \in C(G)^{\mathrm{Ad}\,G}$

- V, W irred $\Rightarrow \langle \chi_V, \chi_W \rangle \triangleq \int \overline{\chi_V} \chi_W = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$

- $\langle \chi_V, \chi_V \rangle = 1 \iff V$ irred.

Proof: $\rho : G \to GL(V)$

$$\int \chi_V(g)\,dg = \int \mathrm{Tr}(\rho(g))\,dg = \mathrm{Tr}\underbrace{\int \rho(g)\,dg}_{p : V \to V}$$

projection to V^G

$$= \dim V^G$$

$$\langle \chi_V, \chi_W \rangle = \int \underbrace{\overline{\chi_V}\,\chi_W}_{\chi_{\mathrm{Hom}(V,W)}} = \dim \mathrm{Hom}(V,W)^G$$

(Assume V, W irred) $= \begin{cases} 1 & \cong \\ 0 & \not\cong \end{cases}$ ($\because$ Schur)

complete reducibility ($V = \bigoplus (\text{irred})$) $\Rightarrow$

- $\|\chi_V\|^2 = 1$ iff irred.
- V det. by χ_V

#

Remark: $G \curvearrowright V$ irred.

$$\Rightarrow \quad \int \chi_V(g^2)\, dg = 1,\ 0,\ -1$$

$[=1]$ real type

i.e. $\exists\ G \curvearrowright V_{\mathbb{R}} / \mathbb{R}$ s.t. $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$

$\Leftrightarrow \exists$ (conjugation) $K : V \longrightarrow \bar{V}$ G-map, $K^2 = 1$

$\Leftrightarrow \exists$ non-degen. $B \in (\mathrm{Sym}^2 V^*)^G$

$$\begin{array}{ccc} & \mathbb{R}^n & O(n) \\ & \downarrow \otimes \mathbb{C} & \downarrow \\ G \curvearrowright & \mathbb{C}^n & U(n) \\ & \| & \downarrow \\ & \mathbb{R}^{2n} & O(2n) \end{array}$$

$[=-1]$ quaternionic type

i.e. $\exists$ (IJK-str) $K : V \longrightarrow \bar{V}$ G-map $K^2 = -1$

$\Leftrightarrow \exists$ non-degen. $B \in (\Lambda^2 V^*)^G$

$$\begin{array}{ccc} & \mathbb{C}^n & U(n) \\ & \downarrow \otimes \mathbb{C} & \downarrow \\ G & \mathbb{H}^n & Sp(n) \\ & \| & \downarrow \\ & \mathbb{C}^{2n} & U(2n) \end{array}$$

$[=0]$ complex type

Remark: Complexification (III 7, 8)

V (vector sp.) $\rightsquigarrow$ V^* $\rightsquigarrow$ $V^{**} \simeq V$

M (topo. sp.) $\rightsquigarrow$ $C(M)$ (alg.) $\rightsquigarrow$ $\{\text{max. ideals in } C(M)\} \simeq M$

G (Lie gp.) $\rightsquigarrow$ $C(G,\mathbb{R})$ (alg. + co-alg) Hopf alg. $\rightsquigarrow$ $\left\{ \begin{array}{c} C(G,\mathbb{R}) \to \mathbb{R} \\ \text{alg. homo.} \end{array} \right\} \simeq G$

Tannaka-Krein duality.

Application: Use $C(G,\mathbb{C})$ to obtain

$G_{\mathbb{C}}$, complexification of G.

§ Peter-Weyl theorem (III 1,2,3,4)

- G finite gp. $\Rightarrow$ $\underbrace{\mathbb{C}[G]}_{L^2(G)} = \bigoplus \mathrm{End}(V_i)$ V_i: irred.

- Fourier Series

$$L^2(S^1) = \overline{\langle \underbrace{e^{2\pi i n\theta}}_{\chi_{V_{irr.}}} \rangle}_{n\in\mathbb{Z}}$$

Compare Finite | Compact Lie

$$\mathbb{C}[G] \supseteq G$$ group ring

$$C(G) \subseteq L^2(G) \supseteq G$$ as δ-fu.

convolution

$$\left(\sum_g \lambda_g g\right) * \left(\sum_h \mu_h h\right) = \sum_{g,h} \lambda_g \mu_h\, g h = \sum_g \left(\sum_h \lambda_{gh^{-1}} \mu_h\right) g$$

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, dh$$

Thm: G compact Lie gp

- $\langle \chi_V \mid V \text{: irred} \rangle$ dense in $C(G)^{\mathrm{Ad}\,G}$

$$\chi_V : G \xrightarrow{\rho} GL(\mathbb{C}^n) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$
$$\hookrightarrow \xrightarrow{a_{ij}} \mathbb{C}$$

"representative functions"
"matrix coeff"

(i.e. $\mathrm{End}\,V \longrightarrow C(G)$
$e_i \otimes e^j \mapsto f$ w/ $f(g) = e_i(g.e^j)$)

- $\langle a_{ij} \mid V\text{: irred} \rangle$ dense in $(C(G), |\cdot|_{\sup})$ & $(L^2(G), \|\cdot\|_{L^2})$

- $L^2(G) = \overline{\bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V)}$

Applications: $G \leq U(N)$

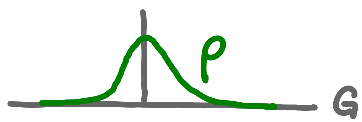
(∵ i) cts fu. separate pt on mfd.
ii) Peter-Weyl $\Rightarrow$ rep. as many as fu.)

$\forall\, G \curvearrowright V_{irr.}$ $\qquad V_{irr} \leq \left(\bigotimes^{k} \mathbb{C}^N\right) \otimes \left(\bigotimes^{\ell} \overline{\mathbb{C}}^N\right)$

Proof of Peter-Weyl thm.:

- $G_L \curvearrowright C(G) \curvearrowleft G_R$ $\qquad f(g^{-1}x)$, $f(xg)$

$$\bigoplus_{U \in Irr(G)} End(U) \cong \left\{\begin{matrix}\text{matrix}\\ \text{coeffs}\end{matrix}\right\} = \left\{ f \in C(G) \;\middle|\; \begin{matrix} f \text{ generate finite dim.} \\ G_L\text{-subsp. in } C(G) \end{matrix} \right\}$$

- Convolution w/ 

$$K : L^2(G) \longrightarrow C(G), \qquad K(f) = f * \rho$$

$$\rho \sim \delta\text{-fu.} \quad \Rightarrow \quad |Kf - f|_{sup} < \varepsilon$$

$K : L^2 \circlearrowleft$ self-adj., compact (K(bound) is precompact)

Functional Analy $\Rightarrow$ Eigenspaces $\bigoplus_{|\lambda| \geq \varepsilon > 0} H_\lambda$ finite dim.

$\bigoplus_\lambda H_\lambda$ dense in L^2.

- $\rho(x) = \rho(x^{-1}) \Rightarrow K$: G-equivar. $\Rightarrow H_\lambda \subseteq L^2(G)$ G-inv.

- fcl analy. $\Rightarrow$ $\bigoplus H_\lambda$ dense in $L^2(G)$

$\Rightarrow$ $\underbrace{\bigoplus \underbrace{K(H_\lambda)}_{\text{finite dim}}}_{\substack{(\because K(H_0)=0) \\ \subseteq \{\text{matrix coeff}\}}}$ dense in $K(L^2(G))$

$\left\{ \begin{array}{l} \varepsilon\text{-close} \\ |\cdot|_{\sup}. \end{array} \right.$

$L^2(G)$

$\square$

§ Weyl Integral formula (IV 1,2)

$T \leqslant G$ torus (Abelian)

$$q : G/T \times T \longrightarrow G$$
$$([g], t) \longmapsto g t g^{-1}$$

SAME dimensions. $\deg q = ?$

Eg. $T = \{e\} \Rightarrow q$ = const map $\Rightarrow \deg q = 0$

$\rightsquigarrow$ Need BIG T

Choose $\overset{\text{generator}}{s} \in T \leqslant G$

$([g], t) \in q^{-1}(s) \iff g t g^{-1} = s \in T$

$\iff$ need $g^{-1} s g \in T$

$\iff g^{-1} T g \subseteq T$ $\quad$ ($\because$ s generator)

i.e. $[g] \in N(T)/T$

If $T \leqslant G$ maximal torus $\implies \underbrace{N(T)/T}_{W}$ finite

Weyl group

Claim: $\deg q = |W|$ $\qquad q : G/T \times T \xrightarrow{g t g^{-1}} G$

i.e. $\det(dq) > 0$

Compute $\det(dq)$ at $([g], t)$

$$
\begin{tikzcd}
T_{[g]}(G/T) \times T_t T \arrow[rr, "dq_{(g,t)}"] & & T_{gtg^{-1}} G \arrow[d, "(l_{gt^{-1}g^{-1}})_*"] \\
\mathfrak{g}/\mathfrak{t} \arrow[u, "l_{g*}"] \times \mathfrak{t} \arrow[u, "l_{t*}"'] & & \mathfrak{g}
\end{tikzcd}
$$

SAME det

$$
\begin{tikzcd}
G/T \times T \arrow[r, "l_g \times l_t"] & G/T \times T \arrow[r, "q"] & G \arrow[r, "l_{gt^{-1}g^{-1}}"] & G
\end{tikzcd}
$$

$$([x], y) \longmapsto (gx, ty) \overset{q}{\longmapsto} (gx)(ty)(gx)^{-1} \longmapsto \underbrace{g t^{-1} x t y x^{-1} g^{-1}}_{\underbrace{\cancel{c(g)}}_{\text{Not affect } \det(dq)}\left(c(t^{-1})(x) \cdot y \cdot x^{-1}\right)}$$

$$\det(dq) \text{ at } ([g], t)$$

$$= \det\Big((\delta x, \delta y) \mapsto \mathrm{Ad}(t^{-1})(\delta x) + \delta y - \delta x \Big)$$

$$= \begin{array}{c} \begin{vmatrix} \mathrm{Ad}(t^{-1}) - I & 0 \\ 0 & I \end{vmatrix} \end{array} \begin{array}{c} \mathfrak{g}/\mathfrak{t} \\ \mathfrak{t} \end{array}$$

$$= \det\left(\mathrm{Ad}(t^{-1}) - I_{\mathfrak{g}/\mathfrak{t}} \right)$$

Can show $\det > 0$ on $q^{-1}(t)$, $t \overset{\text{generator}}{\in T}$

(reason: $\lambda_i = 0 \rightsquigarrow T$ can be made bigger.)

$$\begin{array}{rccc} q : & G/T \times \underset{\text{max torus}}{T} & \longrightarrow & G \\ & ([g], t) & \longmapsto & g t g^{-1} \end{array}$$

$\deg(q) = |W| \neq 0 \Rightarrow q$: surjective

$\Longrightarrow$ (1) T, T' max torus $\Rightarrow T' = g T g^{-1}$, $\exists g$

(pf : choose generator $t' \in T' \subseteq G$

q : surj $\Rightarrow t' \in g T g^{-1} \quad \exists g$

$\Rightarrow T' \subseteq g T g^{-1}$ ($\because$ generator)

$\Rightarrow \quad =$ ($\because$ max.))

(2) Every elt is contained in a max. torus.

$\Downarrow$

(3) $\exp : \mathfrak{g} \to G$ surjective ($\because$ surj for T)

(4) $T = Z(T)$ centralizer

Pf: $x \in Z(T) \setminus T$

$B := \overline{\langle T, x \rangle}$ cpt. Abelian

$\Rightarrow B \cong B_0 \times \mathbb{Z}/m$ (B_0 : conn. comp (torus))

$\Rightarrow B = \overline{\langle g^n \mid n \in \mathbb{Z} \rangle}$ $\exists g \in B$

(2) $\Rightarrow g \in T'$ $\Rightarrow B \subseteq T'$ $\Rightarrow$ "$=$"

(5) $C(G) = \bigcap\limits_{T:\text{max torus}} T$ $\left(\because C(G) \subseteq Z(T) \overset{(4)}{=} T \quad \forall T\right)$

(6). $W \curvearrowright T$ effectively (by (4)).

(7) $G/\mathrm{Ad}(G) \cong T/W$

$R(G) \cong R(T)^W$

Recall $\quad q : G/T \times T \xrightarrow{g t g^{-1}} G$

$$\det(dq)([g], t) = \det(Ad(t^{-1}) - I_{\mathfrak{g}/\mathfrak{t}})$$

Recall: $\mathbb{R}^n \xrightarrow[\text{diffeo.}]{q} \mathbb{R}^n \xrightarrow{f} \mathbb{R} \Rightarrow \int f = \int q^*(f) \cdot \det(dq)$

$$\Rightarrow \int_G f(g)\, dg \qquad \forall f \in C(G)$$

$$= \frac{1}{|W|} \int_T \left[\det(Ad(t^{-1}) - I_{\mathfrak{g}/\mathfrak{t}}) \int_G f(g t g^{-1})\, dg \right] dt$$

$\left(\because \int_T dt = 1\right)$

$$= \frac{1}{|W|} \int_T \det(Ad(t^{-1}) - I_{\mathfrak{g}/\mathfrak{t}})\, f(t)\, dt$$

provided $f \in C(G)^{AdG}$ class function.

§ Weyl Character formula (VI 1,2,3)

$$T = S^1 \leq SU(2) \curvearrowright V = \mathbb{C}^2$$

$$e^{i\theta} \sim \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \rightsquigarrow \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \begin{pmatrix} - \\ - \end{pmatrix}$$

$$\Rightarrow \chi_V\big|_T (e^{i\theta}) = e^{i\theta} + e^{-i\theta} = \frac{e^{2i\theta} - e^{-2i\theta}}{\underbrace{e^{i\theta} - e^{-i\theta}}_{\text{as appeared in Weyl character formula.}}}$$

$$\chi_{S^2V}(e^{i\theta}) = e^{2i\theta} + 1 + e^{-2i\theta} = \frac{e^{3i\theta} - e^{-3i\theta}}{e^{i\theta} - e^{-i\theta}}$$

Similarly, $\chi_{S^nV}(e^{i\theta}) = \dfrac{e^{(n+1)i\theta} - e^{-(n+1)i\theta}}{e^{i\theta} - e^{-i\theta}}$

Fact: S^nV's are ALL irred. repr. of $SU(2)$.

$$\begin{array}{ccc} \mathfrak{t} & & \\ \Big\downarrow{\scriptstyle \exp} & & \\ T & \xrightarrow[\det(\mathrm{Ad}(t^{-1}) - I_{\mathfrak{g}/\mathfrak{t}})]{} & \mathbb{C} \end{array}$$

$(e^{\alpha} = e^{2\pi i \alpha} : \mathfrak{t} \to \mathbb{C})$

$\mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

$$e^{\det(\mathrm{Ad} - I)} = \prod_{\alpha \in R} (e^{\alpha} - 1)$$

$$= \prod_{\alpha \in R_+} (e^{\alpha} - 1)(e^{-\alpha} - 1)$$

$$= \underbrace{\left(\prod_{\alpha \in R_+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \right)}_{\delta} \underbrace{\overline{\left(\prod_{\alpha \in R_+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \right)}}_{\overline{\delta}}$$

$$\delta : \mathfrak{t} \longrightarrow \mathbb{C}$$

$$\delta(H) = \prod_{\alpha \in R_+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$$

$$= e^{\rho(H)} \prod_{\alpha \in R_+} (1 - e^{-\alpha(H)})$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$$

Suppose $G \curvearrowright V$ irred. w/ character $\chi \in C(G)^{\mathrm{Ad}\,G}$

$$1 = \int_G \chi \cdot \overline{\chi}$$

$$= \frac{1}{|W|} \int_T \chi \cdot \overline{\chi} \cdot \det(\mathrm{Ad}\,t^{-1} - I)\, dt$$

$$= \frac{1}{|W|} \int_{\mathfrak{t}} e^{\chi} \cdot \overline{e^{\chi}} \cdot \underbrace{e^{\det(\mathrm{Ad} - I)}}_{\delta \cdot \overline{\delta}}$$

$$\Rightarrow \quad \langle e^{\chi} \cdot \delta ,\ e^{\chi} \cdot \delta \rangle = |W|$$

$$\begin{cases} e^{\chi} : \text{symmetric} \\ \delta : \text{alternating} \end{cases} \quad \text{w.r.t. } W \curvearrowright \mathfrak{t} \qquad \begin{matrix} (\varphi \circ w = \varphi) \\ (\varphi \circ w = (\det w)\varphi) \end{matrix}$$

$$\Rightarrow \quad e^{\chi} \cdot \delta \ \text{ alternating}$$

$$e^{\chi}\cdot\delta$$

- alternating
- $= \sum n_j \, e^{\lambda_j}$

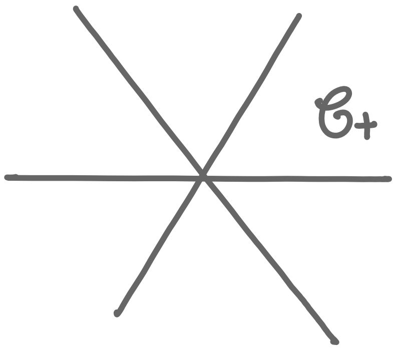
$n_j \in \mathbb{Z}$

$\lambda_j \in \mathfrak{t}^*$

$$\Rightarrow \quad e^{\chi}\cdot\delta = \sum n_j \, A(\gamma_j)$$

$n_j \in \mathbb{Z}$

$\gamma_j \in \mathcal{C}_+ \subset \mathfrak{t}^*$



$$A(\gamma_j)(H) := \sum_{w\in W} \det(w)\, e^{\lambda(wH)}$$

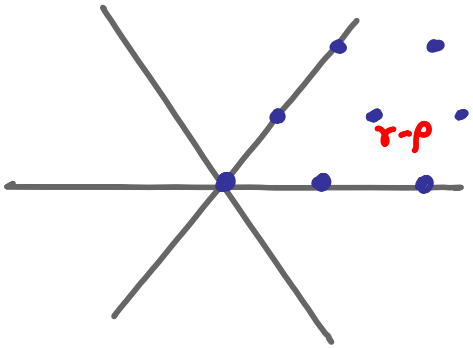
$$\langle A(\gamma), A(\gamma')\rangle = \begin{cases} 0 & \gamma \neq \gamma' \\ |W| & \gamma = \gamma' \end{cases}$$

$$\Rightarrow \quad e^{\chi}\cdot\delta = \pm A(\gamma) \qquad \exists\, \gamma \in \mathcal{C}_+$$

- $A(\gamma)/\delta$ can be decend from $\mathfrak{t}$ to T $\Big\} \Rightarrow \gamma - \rho \in \overline{\mathcal{C}}_+ \cap I^*$

Claim: Conversely, $\gamma-\rho \in \overline{\mathcal{C}_+} \cap I^*$

then $A(\gamma)/\delta = e^{\chi}|_{t}$ $\exists$ irred character.
(or its negative)



Pf: $A(\gamma)/\delta : t \longrightarrow \mathbb{C}$ symm $= \dfrac{\text{alt.}}{\text{alt.}}$

$\Rightarrow$ $A(\gamma)/\delta = f|_T$ $\exists f \in C(G)^{Ad(G)}$.
$\overset{\shortparallel}{C(T)^W}$

Recall: {irred char} : complete o.n. system in $C(G)^{Ad\,G}$.

$$\langle f, \chi \rangle = \int_G f \cdot \overline{\chi} = \frac{1}{|W|} \int_T \frac{Ad(\gamma)}{\delta} \cdot \overline{\chi} \cdot (\det(Ad - I))$$

$$= \int_t \frac{A(\gamma)}{\cancel{\delta}} \cdot \underbrace{\overline{e^{\chi}} \cdot \cancel{\delta} \cdot \overline{\delta}}_{\substack{A(\beta) \\ \exists \beta \in \mathcal{C}_+}} = \pm \int_t A(\gamma) \cdot \overline{A(\beta)} = \begin{cases} 0 \\ \pm |W| \end{cases}$$

$\langle f, \chi \rangle$ can't always zero ($\because \chi$: o.n. system) $\Rightarrow$ DONE.

Theorem: $\overline{\mathscr{C}_+} \cap I^* \longleftrightarrow \mathrm{Irr}(G)|_T$

$$\beta \qquad \qquad \frac{\sum_{w\in W} \det(w)\cdot e^{(\beta+\rho)\cdot w}}{\prod_{\alpha\in R_+} (e^{\alpha/2} - e^{-\alpha/2})}$$

- $\beta = 0 \qquad (V_0 = \mathbb{C})$

$$\sum_{w\in W} \det(w)\cdot e^{\rho\cdot w} = \prod_{\alpha\in R_+} (e^{\alpha/2} - e^{-\alpha/2})$$

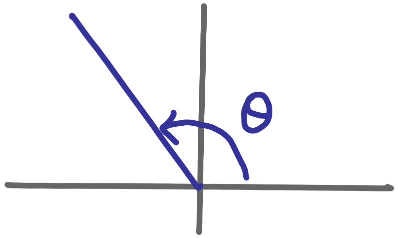
- $\dim V_\beta = \prod_{\alpha\in R_+} \dfrac{\langle \alpha, \beta+\rho\rangle}{\langle \alpha, \rho\rangle}$

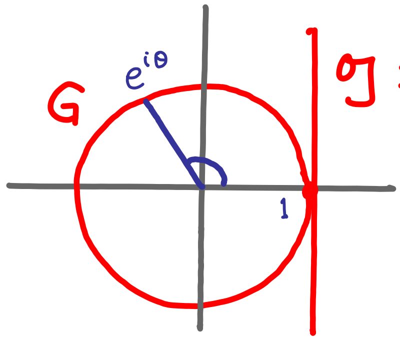
$\left(\text{Pf: } \dfrac{f(0)}{g(0)} = \dfrac{0}{0} \rightsquigarrow \dfrac{f'(0)}{g'(0)} \right)$

Appendix :

Eg. $G = SO(2) \ni$ oriented isometry of $\mathbb{R}^2 \simeq \mathbb{C}$

$= S^1$ $\equiv$ rotation





$\mathfrak{g} = T_1 G \simeq i\mathbb{R}$

$i\theta$

$G \curvearrowright V \simeq \mathbb{C}^r$

irred. $\Rightarrow$ $\rho_n : S^1 \curvearrowright V_n \simeq \mathbb{C}$, $\rho_n(e^{i\theta}) \cdot z = e^{in\theta} z$. $n \in \mathbb{Z}$.

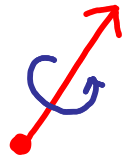
$\chi_n = \mathrm{Tr}(\rho_n) : S^1 \longrightarrow \mathbb{C}$, $e^{i\theta} \mapsto e^{in\theta}$.

$$L^2(S^1)^{Ad(S^1)} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\chi_n$$

$$\parallel$$

$$L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} \mathrm{End}(V_n)$$

- $T = N(T) = G$ ($\because$ Abelian), $W = 1$

Eg. $G = SO(3) \ni$ oriented isometry of $\mathbb{R}^3$ 

$\equiv$ rotation around some axis $\mathbb{R}v$.

w/ angle $\theta \in [0, \pi]$

$$\Rightarrow SO(3) \simeq S^2 \times [0,\pi] \Big/ \begin{smallmatrix} (v,0) \sim (u,0) \\ (v,\pi) \sim (-v,\pi) \end{smallmatrix}$$

$$\overset{\text{polar.}}{\simeq} B^3 / \sim$$ ← antipodal id. on ∂B^3.

$$\simeq \mathbb{RP}^3.$$

$$\mathfrak{g} = \mathbb{R}^3 \curvearrowright \mathbb{R}^3 \text{ is } X \cdot Y = \overset{\text{vector product}}{X \times Y}$$

also same $ad(X)(Y) = [X, Y]$.

$$T = SO(2) \subseteq SO(3) \qquad \left(\begin{array}{c|c} SO(2) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array}\right)$$

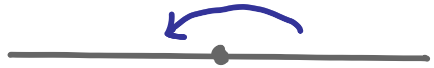
$\ni$ rotation around z-axis.

$$N(T) \ni \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & \pm 1 \end{array}\right) \Rightarrow W = \frac{N(T)}{T} \simeq \mathbb{Z}_2 \text{ or } S_2$$

$$\ni \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & -1 \end{array}\right)$$

$$W \curvearrowright T \qquad \left(\begin{array}{c|c} e^{i\theta} & \\ \hline & 1 \end{array}\right) \mapsto \left(\begin{array}{c|c} e^{-i\theta} & \\ \hline & 1 \end{array}\right)$$

$W \curvearrowright \mathfrak{t} \simeq i\mathbb{R}$



reflection wrt origin (wall)

$$G/Ad(G) \simeq T/W$$

Repr. of $SO(3)$

$$SO(3) \curvearrowright \mathrm{Sym}^{\ell}\, \mathbb{R}^{3*} \ni$$ deg ℓ homog. poly. in x_1, x_2, x_3.

Not irred.

Action commute w/ $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$

$$\Rightarrow SO(3) \curvearrowright \mathrm{Sym}^{\ell}\, \mathbb{R}^{3*} \cap \mathrm{Ker}\,\Delta =: \mathcal{H}_\ell$$

Ex: Write $f(x_1, x_2, x_3) = \displaystyle\sum_{k=0}^{\ell} \frac{x_1^k}{k!} f_k(x_2, x_3) \in \mathrm{Sym}^{\ell}\, \mathbb{R}^{3*}$

$$\Delta f = 0 \iff -f_{k+2} = \frac{\partial^2 f_k}{\partial x_2^2} + \frac{\partial^2 f_k}{\partial x_3^2}$$

$$\therefore \quad \dim \mathcal{H}_\ell = 2\ell + 1.$$